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Modified KdV solitons with non-zero vacuum parameter obtainable from the ZS–AKNS inverse method

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Abstract. We obtain a new N-soliton solution to the modified KdV equation with non-zero vacuum parameter b using the inverse scattering method (see Zakharov and Shabat and Ablowitz *et al*) under a non-vanishing condition (Kawata and Inoue). As demonstrations, we study the special features of the one-soliton and two-solitons in detail. The amplitude, width, and magnitude of the velocity of these solitons are all dependent on b. They all move in the positive direction—such a characteristic is different from that of the KdV solitons.

1. Introduction

Recently, we have obtained new solutions to the Korteweg–de Vries equation $u_t + 12uu_x + u_{xxx} = 0$ using our Bäcklund transformation (Au and Fung 1982, 1984, Wahlquist and Estabrook 1975, Loo Win *et al* 1979) and have discovered that the vacuum parameter b has special physical meaning. We followed up our work by revisiting the inverse scattering theory and have obtained the same one-soliton KdV solution using the generalised inverse scattering method (Au-Yeung *et al* 1983) where $u(x, t) \to b \neq 0$ as $x \to \pm\infty$. To pursue our work further, we intend to solve other nonlinear problems under the non-vanishing boundary condition (i.e. the solution tends to non-zero constant as x tends to $\pm\infty$).

In this paper, we apply the inverse scattering method to find solutions of the modified KdV equation. Along this line of development, Zakharov and Shabat (1972) proposed a two-dimensional inverse scattering theory for the nonlinear Schrödinger equation and obtained solutions having vanishing asymptotic form. Ablowitz, Kaup, Newell and Segur (1973) have generalised the above method and have found the Lax pairs for a number of types of nonlinear equations (the ZS–AKNS inverse method). Previously, the inverse scattering method is used to solve nonlinear equations under vanishing boundary condition. In 1973 Zakharov and Shabat analysed the nonlinear Schrödinger equation under non-vanishing boundary conditions; however, the analytical properties of the associated eigenvalue problem has not been treated. Kawata and Inoue (1977, 1978) made clear the analytical property of the AKNS eigenvalue problem (Ablowitz *et al* 1973) under certain non-vanishing conditions (see § 2.1). We have used the method of Kawata and Inoue to study the modified KdV equation under a non-vanishing boundary condition, and have obtained a new N-soliton solution with non-zero vacuum parameter b. We study both the one-soliton and two-soliton solutions in detail and we find that these solitons propagate only along the positive direction, whatever the values of b. The amplitude, width, and the magnitude of the velocity are all dependent on b.

2. The inverse scattering method for the modified KdV equation under the non-vanishing boundary condition $u \to b$ as $x \to \pm\infty$

2.1. General formalism

Let $u(x, t)$ be a solution of the modified KdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{1}$$

The AKNS eigenvalue problem for the modified KdV equation is

$$v_x = D(\lambda)v, \qquad D(\lambda; x, t) \equiv \begin{pmatrix} -i\lambda & q(x, t) \\ r(x, t) & i\lambda \end{pmatrix}, \tag{2}$$

where

$$q(x, t) \equiv u(x, t) \qquad \text{and} \qquad r(x, t) \equiv -q(x, t). \tag{3}$$

The corresponding time evolution equation is

$$v_t = F(\lambda)v, \qquad F(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} A(\lambda) &\equiv -4i\lambda^3 - 2i\lambda qr + rq_x - qr_x, \\ B(\lambda) &\equiv 4\lambda^2 q + 2i\lambda q_x + 2q^2 r - q_{xx}, \\ C(\lambda) &\equiv 4\lambda^2 r - 2i\lambda r_x + 2qr^2 - r_{xx}. \end{aligned} \tag{5}$$

The matrices $D(\lambda)$ and $F(\lambda)$ satisfy the well known integrable condition of equations (2) and (4):

$$D_t - F_x + DF - FD = 0. \tag{6}$$

In this investigation, we seek for real solution $u(x, t)$ to the modified KdV equation (1) under the following boundary condition:

$$u(x, t) \to b \text{ as } x \to \pm\infty; \qquad b^2 = -\lambda_0^2, \qquad \lambda_0 \text{ is pure imaginary}, \tag{7}$$

and we require that $u(x, t)$ is sufficiently smooth and all the x derivatives of u tend to zero as $x \to \pm\infty$. From (3) and (7), we see that the potentials $q(x, t)$ and $r(x, t)$ satisfy the following non-vanishing condition:

$$q(x, t) \to q^{\pm} \text{ as } x \to \pm\infty, \qquad r(x, t) \to r^{\pm} \text{ as } x \to \pm\infty,$$

and

$$q^+r^+ = q^-r^- = \lambda_0^2. \tag{8}$$

The AKNS eigenvalue problem under the non-vanishing condition (8) is the case considered by Kawata and Inoue (1977, 1978). They have clarified the analytical properties of this problem. Now we apply their method to (2) under condition (8) and obtain a new N-soliton solution with non-zero vacuum parameter b for the modified KdV equation. The details of the derivation are given in § 2.3 and § 2.4. In § 2.2 some of the crucial results of Kawata and Inoue are listed for convenience in our deduction.

2.2. *Some crucial results of Kawata and Inoue*

According to Kawata and Inoue (1977, 1978), the Jost matrices $\Phi^+(\lambda,\zeta)\equiv(\Phi_1^+(\lambda,\zeta),\Phi_2^+(\lambda,\zeta))$ and $\Phi^-(\lambda,\zeta)\equiv(\Phi_1^-(\lambda,\zeta),\Phi_2^-(\lambda,\zeta))$ are defined as the solutions of (2) under the conditions

$$\Phi^+(\lambda,\zeta;x)\to A^+(\lambda,\zeta)\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x}\end{pmatrix} \qquad \text{as } x\to\infty, \tag{9a}$$

$$\Phi^-(\lambda,\zeta;x)\to A^-(\lambda,\zeta)\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x}\end{pmatrix} \qquad \text{as } x\to-\infty, \tag{9b}$$

where

$$\zeta\equiv(\lambda^2-\lambda_0^2)^{1/2}, \tag{10}$$

$$A^+(\lambda,\zeta)\equiv\begin{pmatrix} -iq^+ & \lambda-\zeta \\ \lambda-\zeta & ir^+\end{pmatrix} \qquad \text{and} \qquad A^-(\lambda,\zeta)\equiv\begin{pmatrix} -iq^- & \lambda-\zeta \\ \lambda-\zeta & ir^-\end{pmatrix}. \tag{11}$$

The scattering matrix $S\equiv\left(\begin{smallmatrix} s_{11} & s_{12} \\ s_{21} & s_{22}\end{smallmatrix}\right)$ is defined by the following relation:

$$\Phi^-(\lambda,\zeta;x)=\Phi^+(\lambda,\zeta;x)S(\lambda,\zeta). \tag{12}$$

The function $\zeta=(\lambda^2-\lambda_0^2)^{1/2}$ is a multi-valued function and is set to be single-valued by introducing two Riemann surfaces. When λ_0 is pure imaginary, as in our case of interest (see (7)), a cut is set in the region $(-\lambda_0,\lambda_0)$ and the upper (lower) Riemann surface is defined to be $\zeta\to\lambda$ $(\zeta\to-\lambda)$ as $|\lambda|\to\infty$. The sign of Im ζ is equal (opposite) to the sign of Im λ on the upper (lower) surface.

Kawata and Inoue assume the following representation about the Jost Matrices $\Phi^\pm$:

$$\Phi^+(\lambda,\zeta;x)=A^+(\lambda,\zeta)\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x}\end{pmatrix}+\int_\infty^x K^+(x,y)A^+(\lambda,\zeta)\begin{pmatrix} e^{iy} & 0 \\ 0 & e^{i\zeta y}\end{pmatrix} \tag{13a}$$

$$\Phi^-(\lambda,\zeta;x)=A^-(\lambda,\zeta)\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x}\end{pmatrix}+\int_{-\infty}^x K^-(x,y)A^-(\lambda,\zeta)\begin{pmatrix} e^{-i\zeta y} & 0 \\ 0 & e^{i\zeta y}\end{pmatrix}. \tag{13b}$$

Then functions $K^\pm(x,y)$ satisfy the following three equations:

$$\frac{\partial K^\pm}{\partial x}(x,y)+\sigma_3\frac{\partial K^\pm}{\partial y}(x,y)\sigma_3+\sigma_3K^\pm(x,y)\sigma_3[D^\pm(\lambda)+i\lambda\sigma_3]$$
$$-\begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0\end{pmatrix}K^\pm(x,y)=0, \tag{14a}$$

$$\sigma_3K^\pm(x,x)\sigma_3-K^\pm(x,x)+\Delta D^\pm(\lambda;x,t)=0, \tag{14b}$$

$$K^\pm(x,y)\to0 \qquad \text{as } y\to\pm\infty, \tag{14c}$$

where

$$D^\pm(\lambda)\equiv\begin{pmatrix} -i\lambda & q^\pm \\ r^\pm & i\lambda\end{pmatrix}, \tag{15}$$

$$\Delta D^{\pm}(\lambda; x, t) \equiv D(\lambda; x, t) - D^{\pm}(\lambda) = \begin{pmatrix} 0 & q(x,t)-q^{\pm} \\ r(x,t)-r^{\pm} & 0 \end{pmatrix} \tag{16}$$

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

Assuming the zeros of $S_{11}(\lambda, \zeta; 0)$ in the region Im $\zeta > 0$ are

$$\lambda_j, \qquad \zeta_j = (\lambda_j^2 - \lambda_0^2)^{1/2}, \qquad j = 1, 2, \ldots, N, \tag{18}$$

the Gel'fand–Levitan equation for the function $K^+(x, y)$ is (Kawata and Inoue 1977, 1978)

$$\begin{aligned} K^+(x,y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} &+ \sum_{j=1}^{N} \begin{pmatrix} c_j \\ \tilde{c}_j \end{pmatrix} \exp[\mathrm{i}\zeta_j(x+y)] - H_c(x+y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &- \int_x^\infty K^+(x,y') \sum_{j=1}^{N} \begin{pmatrix} c_j \\ \tilde{c}_j \end{pmatrix} \exp[\mathrm{i}\zeta_j(y+y')]\,\mathrm{d}y' \\ &+ \int_x^\infty K^+(x,y') H_c(y+y') \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}y' = 0, \qquad \text{for } y > x, \end{aligned} \tag{19}$$

where

$$c_j \equiv \tfrac{1}{2}\mathrm{i}(\lambda_j - \zeta_j) m(\lambda_j) \exp[4\mathrm{i}\zeta_j(2\lambda_j^2 + \lambda_0^2)t], \tag{20}$$

$$\tilde{c}_j \equiv -\tfrac{1}{2} r^+ m(\lambda_j) \exp[4\mathrm{i}\zeta_j(2\lambda_j^2 + \lambda_0^2)t], \tag{21}$$

$$m(\lambda) \equiv S_{21}(\lambda, \zeta; 0)/\zeta[\mathrm{d}S_{11}(\lambda, \zeta; 0)/\mathrm{d}\lambda], \tag{22}$$

and $H_c(y)$ is a function such that

$$H_c(y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathrm{i}(\partial h_{1c}/\partial y)(y) + h_{2c}(y) \\ \mathrm{i}r^+ h_{1c}(y) \end{pmatrix}, \tag{23}$$

where

$$h_{1c}(y) \equiv \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\xi y}}{\mu} \left(\frac{S_{21}(\mu, \xi)}{S_{11}(\mu, \xi)} - \frac{S_{21}(-\mu, \xi)}{S_{11}(-\mu, \xi)} \right) \mathrm{d}\xi, \tag{24}$$

$$h_{2c}(y) \equiv \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi y} \left(\frac{S_{21}(\mu, \xi)}{S_{11}(\mu, \xi)} + \frac{S_{21}(-\mu, \xi)}{S_{11}(-\mu, \xi)} \right) \mathrm{d}\xi, \tag{25}$$

$$\mu \equiv (\xi^2 - \lambda_0^2)^{1/2}. \tag{26}$$

They have pointed out that there exist certain symmetries about the AKNS eigenvalue problem under the non-vanishing condition (8). First of all, it has been shown

$$\begin{aligned} S_{11}(\lambda, \zeta) &= (q^-/q^+) S_{22}(\lambda, -\zeta), & S_{11}(\lambda, -\zeta) &= (q^-/q^+) S_{22}(\lambda, \zeta), \\ S_{12}(\lambda, \zeta) &= (-r^-/q^+) S_{21}(\lambda, -\zeta), & S_{12}(\lambda, -\zeta) &= (-r^-/q^+) S_{21}(\lambda, \zeta). \end{aligned} \tag{27}$$

Next, when $r(x, t) = -q(x, t)$, the S matrix has the following additional property:

$$\begin{aligned} S_{11}(\lambda, \zeta) &= S_{22}(-\lambda, -\zeta), & S_{11}(-\lambda, -\zeta) &= S_{22}(\lambda, \zeta), \\ S_{12}(\lambda, \zeta) &= -S_{21}(-\lambda, -\zeta), & S_{12}(-\lambda, -\zeta) &= -S_{21}(\lambda, \zeta), \end{aligned} \tag{28}$$

also, $K^{\pm}(x, y)$ has the following property:

$$K^{\pm}(x, y) = \sigma_{-1} K^{\pm}(x, y) \sigma_{-1}, \tag{29}$$

$$\sigma_{-1} \equiv \begin{pmatrix} 0 & (-1)^{1/2} \\ (-1)^{-1/2} & 0 \end{pmatrix}. \tag{30}$$

Also, when $r(x, t) = -q^*(x, t)$

$$\begin{aligned} S_{11}(\lambda, \zeta) &= S_{22}^*(\lambda^*, \zeta^*), & S_{12}(\lambda, \zeta) &= -S_{21}^*(\lambda^*, \zeta^*), \\ S_{21}(\lambda, \zeta) &= -S_{12}^*(\lambda^*, \zeta^*), & S_{22}(\lambda, \zeta) &= S_{11}^*(\lambda^*, \zeta^*). \end{aligned} \tag{31}$$

2.3. *Relevant results for the modified KdV equation*

In our study, we derive the time dependency of the scattering matrix $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ for the modified KdV equation. Employing equations (2), (4) and (12), we obtain the following result:

$$S(\lambda, \zeta; t) = \exp[-2i\zeta(2\lambda^2 + \lambda_0^2)\sigma_3 t] S(\lambda, \zeta; 0) \exp[2i\zeta(2\lambda^2 + \lambda_0^2)\sigma_3 t], \tag{32a}$$

or

$$\begin{aligned} S_{11}(\lambda. \zeta; t) &= S_{11}(\lambda, \zeta; 0), & S_{12}(\lambda, \zeta; t) &= \exp[-4i\zeta(2\lambda^2 + \lambda_0^2)t] S_{12}(\lambda, \zeta; 0), \\ S_{21}(\lambda, \zeta; t) &= \exp[4i\zeta(2\lambda^2 + \lambda_0^2)t] S_{21}(\lambda, \zeta; 0), & S_{22}(\lambda, \zeta; t) &= S_{22}(\lambda, \zeta; 0). \end{aligned} \tag{32b}$$

From (3), we have $r(x, t) = -q(x, t)$. Also, since $u(x, t)$ is real, we see from (3) that the relation $r(x, t) = -q^*(x, t)$ is also true. Hence relations (28) and (31) are valid for our case of study. Using (22), (27) and (28), we derive the following relations relevant for the modified KdV equation:

$$(q^-/q^+) S_{11}(\lambda, \zeta) = S_{11}(-\lambda, \zeta) \tag{33}$$

and

$$m(-\lambda) = m(\lambda), \tag{34}$$

where λ and $-\lambda$ are in different Riemann sheets. Also, using (28) and (31), we have

$$m(-\lambda^*) = m(\lambda)^*. \tag{35}$$

From (33), we see that if λ is a zero of S_{11} in a Riemann surface, then $-\lambda$ is also a zero of S_{11} in the other Riemann surface. We also see from (35) that $m(\lambda)$ is real if λ is pure imaginary.

2.4. *The N-soliton solution with non-zero vacuum parameter*

We will solve the Gel'fand–Levitan equation (19) for a special situation. Suppose:

(i) the continuous component

$$H_c(y; 0) = 0, \tag{36}$$

(ii) S_{11} has $2N$ zeros in the region $\operatorname{Im} \zeta > 0$, namely, (λ_j, ζ_j) and $(-\lambda_j, \zeta_j)$, $j = 1, \ldots, N$, and λ_j are pure imaginary;

$$\lambda_j = i\eta_j, \qquad \text{where } \eta_j \text{ are positive real constant.} \tag{37}$$

Since $\lambda_0^2=-b^2$ by (7), so we have

$$\zeta_j=\mathrm{i}(\eta_j^2-b^2)^{1/2}. \tag{38}$$

Also, from (34) and (35), we have

$$m(\lambda_j)=m(-\lambda_j) \tag{39}$$

and

$$m(\lambda_j) \text{ is real for } j=1,\ldots,N.$$

Substituting (36), (37) and (39) into (19), (20) and (21), we obtain

$$\begin{aligned}K^+(x,y)\begin{pmatrix}0\\1\end{pmatrix}+\sum_{j=1}^{N}\begin{pmatrix}-\mathrm{i}\zeta_j m(\lambda_j)\exp[4\mathrm{i}\zeta_j(2\lambda_j^2+\lambda_0^2)t]\\-r^+m(\lambda_j)\exp[4\mathrm{i}\zeta_j(2\lambda_j^2+\lambda_0^2)t]\end{pmatrix}\exp[\mathrm{i}\zeta_j(x+y)]\\-\int_x^\infty K^+(x,y')\sum_{j=1}^{N}\begin{pmatrix}-\mathrm{i}\zeta_j m(\lambda_j)\exp[4\mathrm{i}\zeta_j(2\lambda_j^2+\lambda_0^2)t]\\-r^+m(\lambda_j)\exp[4\mathrm{i}\zeta_j(2\lambda_j^2+\lambda_0^2)t]\end{pmatrix}\\\times\exp[\mathrm{i}\zeta_j(y+y')]\,\mathrm{d}y'=0.\end{aligned} \tag{41}$$

Take a representation of $\begin{pmatrix}K_{12}^+\\K_{22}^+\end{pmatrix}$ as

$$\begin{pmatrix}K_{12}^+(x,y)\\K_{22}^+(x,y)\end{pmatrix}=\sum_{j=1}^{N}\begin{pmatrix}K_j(x)\\\tilde{K}_j(x)\end{pmatrix}\exp[\mathrm{i}\zeta_j y], \tag{42}$$

where

$$K^+(x,y)\equiv\begin{pmatrix}K_{11}^+(x,y) & K_{12}^+(x,y)\\K_{21}^+(x,y) & K_{22}^+(x,y)\end{pmatrix}. \tag{43}$$

From (29), we have

$$\begin{pmatrix}K_{11}^+\\K_{21}^+\end{pmatrix}=\begin{pmatrix}K_{22}^+\\-K_{12}^+\end{pmatrix}, \tag{44}$$

hence

$$K^+(x,y)=\begin{pmatrix}K_{22}^+ & K_{12}^+\\-K_{12}^+ & K_{22}^+\end{pmatrix}. \tag{45}$$

Substituting (42) and (45) into (41), we arrive at

$$\begin{aligned}K_j(x)+a_j\exp(\mathrm{i}\zeta_j x)+\sum_{l=1}^{N}\tilde{K}_l(x)a_j\frac{\exp[\mathrm{i}(\zeta_j+\zeta_l)x]}{\mathrm{i}(\zeta_j+\zeta_l)}+\sum_{l=1}^{N}K_l(x)\tilde{a}_j\frac{\exp[\mathrm{i}(\zeta_j+\zeta_l)x]}{\mathrm{i}(\zeta_j+\zeta_l)}=0,\\\tilde{K}_j(x)+\tilde{a}_j\exp(\mathrm{i}\zeta_j x)+\sum_{l=1}^{N}\tilde{K}_l(x)\tilde{a}_j\frac{\exp[\mathrm{i}(\zeta_j+\zeta_l)x]}{\mathrm{i}(\zeta_j+\zeta_l)}-\sum_{l=1}^{N}K_l(x)a_j\frac{\exp[\mathrm{i}(\zeta_j+\zeta_l)x]}{\mathrm{i}(\zeta_j+\zeta_l)}=0,\end{aligned} \tag{41a}$$

where

$$a_j\equiv-\mathrm{i}\zeta_j m(\lambda_j)\exp[4\mathrm{i}\zeta_j(2\lambda_j^2+\lambda_0^2)t] \quad\text{and}\quad \tilde{a}_j\equiv-r^+m(\lambda_j)\exp[4\mathrm{i}\zeta_j(2\lambda_j^2+\lambda_0^2)t]. \tag{46}$$

The determinant of the coefficient matrix is

$$\Delta=\begin{vmatrix}E & B\\-B & E\end{vmatrix}, \tag{47}$$

where

$$E_{jl} = \delta_{jl} + \tilde{a}_j \frac{\exp[i(\zeta_j + \zeta_l)x]}{i(\zeta_j + \zeta_l)} \qquad \text{for } j, l = 1, 2, \ldots, N \tag{48}$$

and

$$B_{jl} = a_j \frac{\exp[i(\zeta_j + \zeta_l)x]}{i(\zeta_j + \zeta_l)} \qquad \text{for } j, l = 1, 2, \ldots, N. \tag{49}$$

The solution of (41a) is given by

$$K_j(x) = -|B_j|/\Delta \qquad \text{and } \tilde{K}_j(x) = -|B_{N+j}|/\Delta \qquad j = 1, \ldots, N,$$

where B_n, with $n = 1, \ldots, N, N+1, \ldots, 2N$, is the matrix obtained by replacing the nth column of the matrix $\begin{pmatrix} E & B \\ -B & E \end{pmatrix}$ by the following column

$$\begin{pmatrix} a_1 \exp(i\zeta_1 x) \\ \vdots \\ a_N \exp(i\zeta_N x) \\ \tilde{a}_1 \exp(i\zeta_1 x) \\ \vdots \\ \tilde{a}_N \exp(i\zeta_N x) \end{pmatrix}. \tag{50}$$

Using (42) and (50), we obtain

$$K_{22}^{+}(x, x) = -\tfrac{1}{2}\, d(\log \Delta)/dx. \tag{51}$$

By using (14), we obtain $u^2(x, t) = b^2 - 2(dK_{22}^{+}(x, x)/dx)$, substituting (51) into this relation, we arrive at

$$u^2(x, t) = b^2 + d^2(\log \Delta)/dx^2. \tag{52}$$

When $N = 1$, we obtain from (37), (38), (46) and (47) that

$$\Delta = 1 \pm \frac{2b}{\eta_1} \exp\{(\eta_1^2 - b^2)^{1/2}[4(2\eta_1^2 + b^2)t - 2x] + \delta\} + \exp\{(n_1^2 - b^2)^{1/2}[8(2\eta_1^2 + b^2)t - 4x] + 2\delta\}, \tag{53}$$

where

$$\delta = \ln[\eta_1 |m(\lambda_1)|/2(\eta_1^2 - b^2)^{1/2}], \tag{54}$$

and the $-$ or $+$ sign in (53) depends on whether $m(\lambda_1)$ is positive or negative respectively. Substituting (53) into (52), we obtain the one-soliton solution to the modified KdV equation

$$u(x, t) = b + \frac{2(\eta_1^2 - b^2)}{b \pm \eta_1 \cosh\{(\eta_1^2 - b^2)^{1/2}[4(2\eta_1^2 + b^2)t - 2x] + \delta\}}. \tag{55}$$

We would remark that when $b = 0$, solution (55) is identical to that obtained via the inverse scattering method under vanishing boundary condition (Wadati 1972).

Next, we will show that when $N=2$ the solution $u(x, t)$ splits into two solitons of similar form to (55) when $|t| \to \infty$. For $N=2$, the determinant Δ is

$$\Delta = \begin{vmatrix} 1+\tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & \tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} & a_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & a_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} \\ \tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & 1+\tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & a_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & a_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \\ -a_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & -a_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} & 1+\tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & \tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} \\ -a_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & -a_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & \tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & 1+\tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \end{vmatrix}. \tag{56}$$

After rearranging the rows and columns of Δ, we obtain

$$\Delta = \begin{vmatrix} 1+\tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & a_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & \tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} & a_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} \\ -a_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & 1+\tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & -a_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} & \tilde{a}_1 \frac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} \\ \tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & a_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & 1+\tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & a_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \\ -a_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & \tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & -a_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & 1+\tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \end{vmatrix}. \tag{56a}$$

Without loss of generality, we assume

$$\eta_1 < \eta_2. \tag{57}$$

In the case $t \to \infty$, consider first the region

$$2(2\eta_2^2+b^2)t - x = \text{constant}. \tag{58}$$

Using (57) and (58), we obtain

$$2(2\eta_1^2+b^2)t - x \to -\infty. \tag{59}$$

With (58) and (59), the asymptotic form of Δ is

$$\Delta \simeq \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & a_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \\ 0 & 0 & -a_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & 1+\tilde{a}_2 \frac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \end{vmatrix}. \tag{60}$$

Substituting (60) into (52), we arrive at

$$u(x, t) \simeq b + \frac{2(\eta_2^2-b^2)}{b \pm \eta_2 \cosh\{(\eta_2^2-b^2)^{1/2}[4(2\eta_2^2+b^2)t-2x]+\delta_2^+\}}, \tag{61}$$

where

$$\delta_2^+ \equiv \ln[\eta_2|m(\lambda_2)|/2(\eta_2^2-b^2)^{1/2}], \tag{62}$$

and the $-$ or $+$ sign in (61) depends on whether $m(\lambda_2)$ is positive or negative respectively. Next, consider the region

$$2(2\eta_1^2+b^2)t-x=\text{constant}. \tag{63}$$

Using (57) and (63), we obtain

$$2(2\eta_2^2+b^2)t-x\to+\infty. \tag{64}$$

With (63) and (64), the asymptotic form of Δ is

$$\Delta\simeq\begin{vmatrix} 1+\tilde{a}_1\dfrac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & a_1\dfrac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & \tilde{a}_i\dfrac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} & a_1\dfrac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} \\ -a_1\dfrac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & 1+\tilde{a}_1\dfrac{e^{i(\zeta_1+\zeta_1)x}}{i(\zeta_1+\zeta_1)} & -a_1\dfrac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} & \tilde{a}_1\dfrac{e^{i(\zeta_1+\zeta_2)x}}{i(\zeta_1+\zeta_2)} \\ \tilde{a}_2\dfrac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & a_2\dfrac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & \tilde{a}_2\dfrac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & a_2\dfrac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \\ -a_2\dfrac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & \tilde{a}_2\dfrac{e^{i(\zeta_2+\zeta_1)x}}{i(\zeta_2+\zeta_1)} & -a_2\dfrac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} & \tilde{a}_2\dfrac{e^{i(\zeta_2+\zeta_2)x}}{i(\zeta_2+\zeta_2)} \end{vmatrix}. \tag{65}$$

Substituting (37), (38) and (46) into (65), we have

$$\begin{aligned}\Delta=\{&1\pm(2b/\eta_1)\exp[(\eta_1^2-b^2)^{1/2}][4(2\eta_1^2+b^2)t-2x]+\delta_1^+\\ &+\exp[(\eta_1^2-b^2)^{1/2}][8(2\eta_1^2+b^2)t-4x]+2\delta_1^+\}\\ &\times\exp(4i\zeta_2x)\begin{vmatrix}\dfrac{\tilde{a}_2}{i(\zeta_2+\zeta_2)} & \dfrac{a_2}{i(\zeta_2+\zeta_2)}\\ \dfrac{-a_2}{i(\zeta_2+\zeta_2)} & \dfrac{\tilde{a}_2}{i(\zeta_2+\zeta_2)}\end{vmatrix},\end{aligned} \tag{66}$$

where

$$\delta_1^+=\ln\frac{\eta_1|m(\lambda_1)|}{2(\eta_1^2-b^2)^{1/2}}+2\ln\frac{(\eta_2^2-b^2)^{1/2}-(\eta_1^2-b^2)^{1/2}}{(\eta_2^2-b^2)^{1/2}+(\eta_1^2-b^2)^{1/2}}. \tag{67}$$

Substituting (66) into (52), we arrive at

$$u(x,t)\simeq b+\frac{2(\eta_1^2-b^2)}{b\pm\eta_1\cosh\{(\eta_1^2-b^2)^{1/2}[4(2\eta_1^2+b^2)t-2x]+\delta_1^+\}}. \tag{68}$$

Combining (61) and (68), we arrive at

$$u(x,t)\simeq b+\sum_{n=1}^{2}\frac{2(\eta_n^2-b^2)}{b\pm\eta_n\cosh\{(\eta_n^2-b^2)^{1/2}[4(2\eta_n^2+b^2)t-2x]+\delta_n^+\}} \tag{69}$$

as $t\to\infty$.

By a similar argument, we obtain when $t\to-\infty$,

$$u(x,t)\simeq b+\sum_{n=1}^{2}\frac{2(\eta_n^2-b^2)}{b\pm\eta_n\cosh\{(\eta_n^2-b^2)^{1/2}[4(2\eta_n^2+b^2)t-2x]+\delta_n^-\}} \tag{70}$$

where

$$\delta_1^- \equiv \ln \frac{\eta_1 |m(\lambda_1)|}{2(\eta_1^2 - b^2)^{1/2}}$$

and

$$\delta_2^- = \ln \frac{\eta_2 |m(\lambda_2)|}{2(\eta_2^2 - b^2)^{1/2}} + 2 \ln \frac{(\eta_2^2 - b^2)^{1/2} - (\eta_1^2 - b^2)^{1/2}}{(\eta_2^2 - b^2)^{1/2} + (\eta_1^2 - b^2)^{1/2}}, \tag{71}$$

and the choice of −ve or +ve sign in (70) depends on whether $m(\lambda_n)$ is positive or negative respectively.

From equations (62), (67) and (71), we see that the total phase shift for soliton one after collision is

$$\begin{aligned}\delta_1 &= \delta_1^+ - \delta_1^- \\ &= 2 \ln \frac{(\eta_2^2 - b^2)^{1/2} - (\eta_1^2 - b^2)^{1/2}}{(\eta_2^2 - b^2)^{1/2} + (\eta_1^2 - b^2)^{1/2}},\end{aligned} \tag{72}$$

and the total phase shift for soliton two after collision is

$$\begin{aligned}\delta_2 &= \delta_2^+ - \delta_2^- \\ &= -2 \ln \frac{(\eta_2^2 - b^2)^{1/2} - (\eta_1^2 - b^2)^{1/2}}{(\eta_2^2 - b^2)^{1/2} + (\eta_1^2 - b^2)^{1/2}}.\end{aligned} \tag{73}$$

From (72) and (73), we see that soliton one shifts backwards after collision by an amount $|\Delta x_1|$ where

$$\Delta x_1 = \frac{1}{(\eta_1^2 - b^2)^{1/2}} \ln \frac{(\eta_2^2 - b^2)^{1/2} - (\eta_1^2 - b^2)^{1/2}}{(\eta_2^2 - b^2)^{1/2} + (\eta_1^2 - b^2)^{1/2}}, \tag{74}$$

and soliton 2 shifts forwards after collision by an amount Δx_2 where

$$\Delta x_2 = \frac{-1}{(\eta_2^2 - b^2)^{1/2}} \ln \frac{(\eta_2^2 - b^2)^{1/2} - (\eta_1^2 - b^2)^{1/2}}{(\eta_2^2 - b^2)^{1/2} + (\eta_1^2 - b^2)^{1/2}}. \tag{75}$$

We would remark again that when $b = 0$ the two-soliton solution is identical to that obtained via the inverse scattering method under vanishing boundary condition (Wadati 1972). We see from (74) and (75) that the slower soliton shifts backwards while the faster soliton shifts forwards after collision. Also, we see from (72) and (73) that the total phase is conserved before and after the collision. This result is in general similar to that of the case under vanishing boundary condition. However, the vacuum parameter b modifies the velocity, amplitude, width and the total phase shift of the solitons.

3. Conclusions

(i) Our group of research workers have recently made an effort to find solutions for nonlinear equations with non-zero vacuum parameter b (Au and Fung 1982, Fung and Au 1982), via the Bäcklund transformation. Along this line of thought, we need also to study the solutions of the same types of nonlinear equations using other methods, such as the inverse scattering method under certain non-vanishing boundary conditions.

We have just obtained solutions to the KdV equation under one such boundary condition using the generalised inverse scattering method (Au-Yeung *et al* 1984). To follow our sequence of research, we have presented new solutions to the modified KdV equation with non-zero vacuum parameter in this article. The results of these two articles by the same authors indicate that we have to study solutions outside the square integrable range.

(ii) Based on our solutions (55), the velocity of the Modified KdV one-soliton is

$$v = 2(2\eta^2 + b^2), \tag{76}$$

where η is a positive real parameter (relation (37)). The amplitude is

$$A = \begin{cases} |2\eta - 2b| & \text{if } m(\mathrm{i}\eta) \text{ is negative} \\ |-2\eta - 2b| & \text{if } m(\mathrm{i}\eta) \text{ is positive}' \end{cases} \tag{77}$$

while the width of the soliton is

$$D = \begin{cases} \dfrac{1}{(\eta^2 - b^2)^{1/2}} \ln\left[\dfrac{2\eta + b}{\eta} + \left(\dfrac{(2\eta + b)^2}{\eta^2} - 1\right)^{1/2}\right] & \text{if } m(\mathrm{i}\eta) \text{ is negative} \\ \dfrac{1}{(\eta^2 - b^2)^{1/2}} \ln\left[\dfrac{2\eta - b}{\eta} + \left(\dfrac{(2\eta - b)^2}{\eta^2} - 1\right)^{1/2}\right] & \text{if } m(\mathrm{i}\eta) \text{ is positive.} \end{cases} \tag{78}$$

(iii) The one-soliton KdV solution (55) with non-zero parameter b is new. The velocity, amplitude and width are all generally dependent on b. The propagation velocity of the soliton is positive definite, whatever the values of b are. In other words, the velocity is uni-directional, being different from the property of the KdV one-soliton with non-zero b (Au and Fung 1982). We would like to remark also that Fung and Au (1982) have obtained another set of solutions to the modified KdV equation (see equation (14*c*) taking $\lambda = 0$) and this solution is also uni-directional, namely, the direction of the velocity is independent of b, although the magnitude depends on b.

(iv) To further our research, we have established the general formalism for the N-soliton solution with non-zero vacuum parameter. As a demonstration, we deduce that the two-soliton solution splits apart into two solitons when $|t| \to \infty$. The velocity, the amplitude, the width, the phase shifts and displacements of the two-solitons are all controlled by the vacuum parameter b. This parameter is thus a physically significant quantity. If b takes zero value, our relevant two-soliton solution reduces to that obtained by Wadati (1972).

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